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Uncertainty Inequality for the Entanglement of Both Two-rebits and Two-qubits States

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Abstract
We assume that both the concurrence C_r of a two-rebits state and the concurrence C_q of the usual two-qubits states are represented by hermitian operators (observables). We calculate the respective uncertainty ΔC_r and the uncertainty ΔC_q measured both as the standard deviation . We make the strictly mathematical assumption that there exists a canonical conjugate variable (called ξ) to the concurrence (C) such that both quantities satisfy a Robertson's [1] uncertainty inequality of the form $(\Delta A)^2(\Delta B)^2 > |\frac{1}{2}\langle[A, B]\rangle|^2$. From such inequality we impose bounds for both uncertainties $\Delta\xi_r$ and $\Delta\xi_q$.

Keywords Entanglement · Rebits · Qubits · Uncertainty

Entanglement distinguishes substantially quantum mechanics from classical mechanics [2]. Quantum entanglement plays a key role in superdense coding [3], quantum teleportation [4], and quantum information [5–8]. For a two qubits system (in pure and mixed states), entanglement has been intensively studied in the past [9–13]. Concurrence can be considered as a measure of entanglement of a two-qubits state [9]. In the present work it is studied the uncertainty in the concurrence of both two-rebits and two qubits. In the literature it has not been considered uncertainty for a hypothetical conjugate variable to the concurrence [14, 15].

For quantum mechanics defined over vector real spaces the simplest state is the rebit state which is defined as

$$|\eta\rangle = a_0|0\rangle + a_1|1\rangle, \tag{1}$$

where a_i $(i = 0, 1)$ are real numbers such that $a_0^2 + a_1^2 = 1$ and $\{|0\rangle, |1\rangle\}$ are the classical bits. It is worth mentioning that for an arbitrary two-rebits system in Ref. [16] it was found an expression for the entanglement of formation. Pure states of two-rebits systems are described by 9 quantities. On the other hand, the usual one-qubit state is defined as

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle, \tag{2}$$

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where c_i $(i = 0, 1)$ are a complex numbers such that $|c_0|^2 + |c_1|^2 = 1$ and $\{|0\rangle, |1\rangle\}$ are the basic qubits. By the way, the corresponding space of for a two-qubits state is 15-dimensional.

In the present work we calculate the uncertainty of the concurrence ΔC associated to a pure both two-rebits and two-qubits states. For the above we assume that the concurrence is represented by a hermitian operator (observable). Furthermore, we make a strict mathematical assumption that there exists a hermitian canonical conjugate variable, denoted by ξ, such that this satisfies a Robertson's uncertainty relation [1] of the form

$$\Delta C \Delta \xi \geq \frac{1}{2}|\langle [C, \xi] \rangle| = \frac{1}{2}. \tag{3}$$

Let us observe that the above equation might indicate the existence of a hidden 'uncertainty principle' that would involve the concurrence C and a hypothetical complementary quantity ξ ($\hbar = 1$).

For a two qubits system, entanglement is an associated intrinsically quantum quantity. This has been very well studied in the past. It is well known that concurrence of a two-qubit system is a good measure of the entanglement of formation $E_q(\rho)$ [9] which is defined as

$$E_q[\rho] = h\left(\frac{1 + \sqrt{1 - C_q^2}}{2}\right), \tag{4}$$

wheres

$$h(x) = -x \log_2 x - (1 - x)\log_2(1 - x). \tag{5}$$

The concurrence for a pure two-qubits states is given by

$$C_q = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4), \tag{6}$$

being λ_i, $(i = 1, ..., 4)$ the square root in decreasing order of the eigenvalues of the matrix $\rho\tilde{\rho}$, with

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y). \tag{7}$$

For a two-qubits system the states become less entangled as the degree of mixtures increases. Here restrict ourselves to a pure two-qubit states.

For a two-rebits system in Ref. [16] it was found a formula for the entanglement of formation of two-rebits state which is given by

$$E[\rho_r] = h\left(\frac{1 + \sqrt{1 - C_r^2}}{2}\right), \tag{8}$$

where

$$h(x) = -x \log_2 x - (1 - x)\log_2(1 - x). \tag{9}$$

The above expressions are identical to the case of two-qubits except that the concurrence of a two-rebits is

$$C[\rho_r] = |\mathrm{tr}(\rho_r \sigma_y \otimes \sigma_y)|. \tag{10}$$

For a two-rebits state the entanglement of formation is the expectation value of one single observable $\sigma_y \otimes \sigma_y$ A pure two-rebits state can be writen as

$$|\Psi_r\rangle = \sum_{i=1}^{4} a_i |\phi_i\rangle, \tag{11}$$

where we have used decimal notation, a_i are real numbers such that

$$\sum_{i=1}^{4} a_i^2 = 1, a_i \in \mathfrak{R}. \tag{12}$$

The states $|\phi_i\rangle$ are eigenstates of the operator $\sigma_y \otimes \sigma_y$. In Ref. [17] it was found an analytical expression for the probability density $P(C_r^2)$ of finding a pure two-rebits state with a squared concurrence C_r^2. This has the form

$$P(C_r^2) = \frac{1}{2\sqrt{C_r^2}}. \tag{13}$$

The above expression has a strong divergence at $C_r^2 = 0$. The domain of (13) is $0 \leq C_r^2 \leq 1$. Concerning to a pure two-qubit state, the distribution of probability has been found analytically in [18] and it is

$$P(C_q^2) = \frac{3}{2}\sqrt{1 - C_q^2}. \tag{14}$$

We note that the densities of probability (13) and (14) for two-rebits and two-qubits respectively are even functions of the concurrences. That is, they admit negative values of the concurrences, that is, $P((-C_r)^2) = P(C_r^2)$ and $P((-C_q)^2) = P(C_q^2)$. A negative concurrence has a non trivial interpretation. By the way, the discovery that quantum knowledge can be negative was made by Horodecki-Oppenheim-Winter [19]. They argue that in the quantum world there are things we just cannot know. For instance, we cannot know both the position (energy) and momentum (time) of a microscopic quantum system.

The uncertainty of a variable X is defined as its standard deviation, that is

$$\Delta X = \sqrt{< X^2 > - < X >^2}, \tag{15}$$

where the average value of a function $f(X)$ is

$$< f(X) >= \int_0^1 f(X)P(X)dX, \tag{16}$$

being $P(X)$ the distribution of probability of the variable X.

– *Uncertainty for the concurrence of a two-rebits states* (ΔC_r)

Rebits are a powerful tool for Quantum Information Processing. For instance in Ref. [20] it was concluded that rebits do not affect universality for quantum computation. Rebits may also be defined as a real density matrices of n two-level systems.

By using (13) it is possible to define the expectation value of C_r as

$$\begin{aligned} < C_r > &:= \int_0^1 \sqrt{C_r^2}P(C_r^2)d(C_r^2) \\ &= \int_0^1 \sqrt{C_r^2}\frac{1}{2\sqrt{C_r^2}}d(C_r^2) \\ &= \frac{1}{2}. \end{aligned} \tag{17}$$

The respective expectation value of C_r^2 is

$$< C_r^2 > = \int_0^1 C_r^2 P(C_r^2) d(C_r^2) = \int_0^1 C_r^2 \frac{1}{2\sqrt{C_r^2}} d(C_r^2) = \frac{1}{3}. \quad (18)$$

Substituting (17) and (18) in (15) the uncertainty for the concurrence for a pure two-rebits state is

$$\Delta C_r = \frac{\sqrt{12}}{12} = 0.29. \quad (19)$$

If we assume that the uncertainty ΔC_r satisfies an uncertainty principle such as (3) then the uncertainty of its complementary variable ξ_r satisfies

$$\Delta \xi_r \geq \frac{\sqrt{12}}{2} = 1.7. \quad (20)$$

To interpret ξ_r is a nontrivial task. In fact, this is a challenge for Quantum Information Theory.

– *Uncertainty for the concurrence of a two-qubits states* (ΔC_q)

By using (14) the expectation value of C_q is

$$< C_q > := \int_0^1 \sqrt{C_q^2} P(C_q^2) d(C_q^2) = \frac{3\pi}{16}. \quad (21)$$

The respective expectation value of C_q^2 is

$$< C_q^2 > = \int_0^1 C_q^2 P(C_q^2) d(C_q^2) = \frac{2}{5}. \quad (22)$$

Substituting (21) and (22) in (15) the uncertainty for the concurrence for a pure two-qubits state is

$$\Delta C_q = \sqrt{\frac{2}{5} - \frac{9\pi^2}{256}} = 0.23 \quad (23)$$

If we assume that there exists a hidden uncertainty principle as (3), the uncertainty in the respective complementary variable ξ_q is

$$\Delta \xi_q \geq \frac{1}{2\sqrt{\frac{2}{5} - \frac{9\pi^2}{256}}} = 2.2. \quad (24)$$

The interpretation of the variable ξ_q would remain also as an open question for Quantum Information Science.

We have calculated the uncertainty of the concurrence associated to a pure two-rebits state and also to a pure two-qubits state. Both concurrences have a non zero value which means that their measure cannot be done with precision. This confirms that they are a quantum mechanics quantities that cannot be known with absolute certainty. We observe that both uncertainties have a similar values i.e. $\Delta C_r = 0.29$ and $\Delta C_q = 0.23$. To assume from the point of view strictly mathematical that there exists a complementary quantity ξ to the concurrence C in such a way that both variables satisfy an uncertainty principle (3) is a mere conjecture that requires many efforts and intense search. By choosing a system of units where $\hbar = 1$ it is concluded that the complementary variable ξ of (3) is dimensionless. A future challenge is to identify the hermitian operator ξ with a physical quantity.

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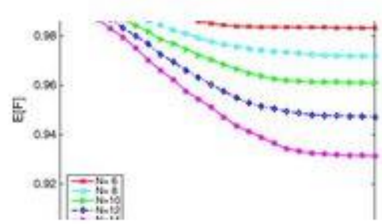

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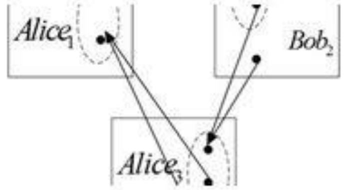

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